MICHIGAN STATE
UNIVERSITY

Name: ____________________
NetID: ____________________

Algebra Qualifying Exam II

August 2019

Instructions:

- Read each problem carefully.
- Write legibly and in complete sentences.
- Cross off anything you do not wish graded.
- This exam has 7 pages with 6 questions, for a total of 60 points. Make sure that all pages are included.
- You may not use books, notes, or electronic devices.
- You may ask proctors questions to clarify problems on the exam.
- Graded work should only be written in the provided spaces (not on scratch paper) unless otherwise approved by a proctor.
- You have 120 minutes to complete this exam.

Good luck!

1. (10 points) Let $A = \mathbb{C}[x]$ be the ring of polynomials over $\mathbb{C}$. Consider a 3×3 matrix

$$M = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

Let $V = \mathbb{C}^3$ be a vector space of dimension 3. We view V as an A-module such that

$$\forall f(x) \in A, \quad \forall \vec{v} \in V, \qquad f(x)\vec{v} := f(M)\vec{v}$$

Prove that V is a torsion module over A, and

$$V \cong A/\left((x-2)^2(x-3)\right)$$

2. (10 points) Prove **Hilbert's basis Theorem**: if R is a commutative Noetherian ring, then the polynomial ring $R[X]$ is Noetherian.

3. (10 points) Consider the polynomial

$$f(x) = 5x^5 + 6x^3 + 6x + 21.$$

Let $\alpha \in \mathbb{C}$ be a complex number such that $f(\alpha) = 0$.

(a) Prove that the polynomial $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) Let $\beta = \sqrt[16]{2}$. Prove that $\alpha \notin \mathbb{Q}[\beta]$.

4. (10 points) Let $a \in \mathbb{Q}$. Prove that the number $\sin(a\pi)$ is algebraic over $\mathbb{Q}$.

5. (10 points) Determine the Galois group of $\mathbb{Q}[\sqrt{3}, \sqrt{5}]$ over $\mathbb{Q}$.

6. (10 points) Let K be a finite field consisting of q many elements, and let $f \in K[x]$ be an irreducible polynomial. Prove that f divides $x^{q^n} - x$ if any only if $\mathbf{deg} f$ divides n.